

# UNIFORM EXCLUSION OF LANDAU–SIEGEL ZEROS

OPENAI (WITH COMMUNITY CONTRIBUTIONS)

**ABSTRACT.** We prove the uniform exclusion of Landau–Siegel zeros. There is an absolute constant  $c > 0$  such that every real zero  $\beta \in (0, 1)$  of every primitive nonprincipal real Dirichlet  $L$ -function of conductor  $q \geq 3$  satisfies  $(1 - \beta) \log q \geq c$ .

## 1. INTRODUCTION

For a primitive Dirichlet character  $\chi$  of conductor  $q$ , let

$$L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s} \quad (\operatorname{Re} s > 1),$$

and use its analytic continuation elsewhere. For an absolute constant  $c_1 > 0$ , the classical region

$$\operatorname{Re} s \geq 1 - \frac{c_1}{\log(q(|\operatorname{Im} s| + 2))}$$

is zero-free except for a possible simple real zero attached to a real character [5, Section 14]. Excluding this possibility uniformly is the Landau–Siegel zero problem. We prove its logarithmic formulation.

**Theorem 1.** *There is an absolute constant  $c > 0$  such that every real zero  $\beta \in (0, 1)$  of every primitive nonprincipal real Dirichlet  $L$ -function of conductor  $q \geq 3$  satisfies*

$$(1.1) \quad (1 - \beta) \log q \geq c.$$

Possible exceptional zeros obstruct uniform estimates for primes in arithmetic progressions. They also affect the study of quadratic class numbers through the values  $L(1, \chi)$  in Dirichlet’s class-number formula. Siegel’s theorem of 1935 gives  $L(1, \chi) \gg_\varepsilon q^{-\varepsilon}$  for every  $\varepsilon > 0$ , with an ineffective constant [18, 5]. This estimate leaves open the possibility of a sequence of real zeros with  $(1 - \beta) \log q \rightarrow 0$ . More uniformly across characters, Page’s theorem gives an absolute constant  $c_2 > 0$  such that, for every  $Q \geq 3$ , at most one primitive real character of conductor at most  $Q$  has a real zero in  $(1 - c_2/\log Q, 1)$  [15]. The common window depends on  $Q$ , not on each character’s own conductor. The Deuring–Heilbronn phenomenon, developed quantitatively by Linnik and Heath-Brown, forces other zeros away from 1 [12, 9]; see Benli, Goel, Twiss, and Zaman [3] for a recent explicit form. Friedlander and Iwaniec [7] discuss the logarithmic zero-gap conjecture and its relation to lower bounds for  $L(1, \chi)$ .

**The argument.** The analytic starting point is that a zero with  $(1 - \beta) \log q$  small forces a shortage of primes with  $\chi(p) = 1$ ; compare [7, Section 5]. Consequently, primes with  $\chi(p) = -1$  carry enough logarithmic mass for the determinant comparison. We give the short logarithmic-derivative proof in Section 2.

The remaining argument is algebraic. Associate to  $\chi$  a quadratic field  $\mathbb{Q}(\sqrt{d})$  and adjoin  $\sqrt{2}$ . Except for one fixed quadratic field, this gives a biquadratic field  $K = \mathbb{Q}(a, b)$ , with  $a^2 = d$  and  $b^2 = 2$ . Let  $\sigma$  change the sign of  $a$  and let  $\tau$  change the sign of  $b$ . For  $n = (n_1, n_2, n_3, n_4) \in \{0, \dots, N - 1\}^4$ , put

$$\theta_n = n_1 + n_2 a + n_3 b + n_4 ab.$$

---

*Date:* October 19, 2026.

*2020 Mathematics Subject Classification.* Primary 11M20; Secondary 11J81, 14L17.

*Key words and phrases.* Dirichlet  $L$ -functions, Landau–Siegel zeros, polynomial interpolation, interpolation determinants.

We form rows indexed by triples  $\alpha \in \mathbb{Z}_{\geq 0}^3$  whose entries are

$$\theta_n^{\alpha_1} \sigma(\theta_n)^{\alpha_2} \sigma\tau(\theta_n)^{\alpha_3}.$$

There are  $N^4$  columns. The interpolation estimate supplies enough independent rows to form a nonzero square determinant at the natural degree scale  $U = N^{4/3}$ , for which  $U^3 = N^4$ . To favor the first exponent, we order the rows by  $\alpha_1 + H\alpha_2 + H\alpha_3$  and retain a row exactly when it increases the span of its predecessors. Here the integer  $H$  will be large and fixed. Summed over the retained rows, the last two exponents account for an arbitrarily small fraction of the total degree.

At an odd prime  $p \nmid q$  with  $\chi(p) = -1$ , taking  $p$ -th powers modulo  $p$  sends  $\theta_n$  to either  $\sigma(\theta_n)$  or  $\sigma\tau(\theta_n)$ . When  $p > H$ , replacing a block of  $p$  first factors by the corresponding conjugate strictly lowers the row weight. Subtraction therefore preserves the determinant and extracts powers of  $p$  from the rows. Primes up to  $U$  contribute the leading factor  $\log U$  to the divisibility bound, whereas the evaluation points contribute  $\log N = \frac{3}{4} \log U$  to the size bound. This strict difference drives the contradiction. Taking  $N$  to be a sufficiently large fixed power of  $q$  controls the additional  $\log q$  term in the size bound. Section 4 constructs the rows, Section 5 proves the two bounds, and Section 6 makes the parameter choices.

**Interpolation and arithmetic context.** The reusable algebraic input is Lemma 3, proved in Section 3. It bounds the separate degrees needed to interpolate on a linear image of an integer box in  $\mathbb{C}^4$ . Its hypothesis is that the kernel is spanned by a vector whose coordinates are linearly independent over  $\mathbb{Q}$ . The estimate is uniform in the linear map and in the choice of coordinates on the three-dimensional image. This allows the interpolation box to have an arbitrarily large fixed aspect ratio without introducing a constant depending on the field. The proof uses a dimension count, a nearest-point argument for an integer polytope, and Lagrange interpolation on a supporting face.

General multiplicity estimates on commutative algebraic groups are developed in [17]; rectangular derivative conditions and interpolation duality appear in [6, Theorem 1.1 and Lemma 1.3]. The estimate proved here makes the coefficient and coordinate uniformity explicit. The final comparison belongs to the interpolation-determinant method described by Laurent [11, Section 6]. Bost's arithmetic algebraicity criteria use spaces of derivations closed under  $p$ -fold composition at almost all prime ideals [4, Theorems 2.1 and 2.3]. In the torus interpretation recalled in Section 3, our argument controls this operation on one distinguished derivation at selected primes and uses it directly for determinant divisibility, rather than applying an algebraicity criterion. All algebraic estimates used in the proof are established below.

*Remark* (Subsequent expositions). Several expositions and variants of the argument of this paper have since appeared. Banks [2] gives a detailed direct proof with explicit constants: for conductors  $q \geq 10^{10}$  one may take  $c = 3.0 \times 10^{-3}$ , and combined with the numerical verification of Lu, Zaman, and Zhao [13] this covers every conductor  $q \geq 3$ . Kerr [10] works in the same biquadratic field  $\mathbb{Q}(\sqrt{d}, \sqrt{2})$  as here, but replaces the interpolation lemma and the determinant analysis by a pigeonhole construction of a polynomial of bounded degree vanishing at the relevant linear forms, which is then contradicted by a zero estimate of Masser [14]. Green [8] gives a variant that works entirely in the quadratic field  $\mathbb{Q}(\sqrt{d})$  attached to  $\chi$ , avoiding the passage to a biquadratic field. There the interpolation determinant is replaced by the explicit Gram determinant of a family of evaluation vectors, whose archimedean size and  $p$ -adic valuations at primes with  $\chi(p) = -1$  are compared directly. A sketch of Green's argument is given by Tao [19], who also records an observation of A. Pascadi: adjoining further square roots in biquadratic arguments of the type given here brings the coefficient of  $\log x$  in the resulting upper bound for  $\sum_{p \leq x, \chi(p)=-1} (\log p)/p$  arbitrarily close to  $1/2$ . Further variants are due to Pescador [16] and Barreto [1], who sharpen this bound to  $(1/2) \log x + (1/4) \log q + \log \log x + O(1)$ . Barreto's three-page argument establishes the nonsingularity of the relevant matrix using a Wronskian in place of resultants, and deduces  $(1 - \beta) \log q \geq 1 - O(\log \log q / \log q)$ .

All logarithms are natural. Constants implicit in  $O(\cdot)$  and  $\ll$  are absolute unless a dependence is specified. All vector-space spans are over the indicated field, and we use  $z^0 = 1$ , including when  $z = 0$ .

## 2. THE PRIME BIAS FORCED BY A REAL ZERO

We first record the prime-sum estimate that will be compared with the determinant bound. Let  $\chi$  be primitive, nonprincipal, and real of conductor  $q \geq 3$ , and let  $\beta \in (0, 1)$  be a real zero. Write

$$\ell = \log q, \quad \delta = (1 - \beta)\ell.$$

**Lemma 2.** *For  $X \geq 3$ ,*

$$(2.1) \quad \sum_{\substack{p \leq X \\ \chi(p)=1}} \frac{\log p}{p} \ll \ell + \frac{\delta(\log X)^2}{\ell}.$$

*Consequently, for each integer  $H \geq 2$ ,*

$$(2.2) \quad \sum_{\substack{H < p \leq X \\ p \nmid 2q \\ \chi(p)=-1}} \frac{\log p}{p} \geq \log X - C\ell - C \frac{\delta(\log X)^2}{\ell} - C_H,$$

*where  $C$  is absolute and  $C_H$  depends only on  $H$ .*

*Proof.* Put  $\varepsilon = (1 - \chi(-1))/2$ . For the completed function

$$\Lambda_\chi(s) = (q/\pi)^{(s+\varepsilon)/2} \Gamma((s+\varepsilon)/2) L(s, \chi),$$

the Hadamard product and functional equation give, for real  $s > 1$ ,

$$(2.3) \quad -\frac{L'}{L}(s, \chi) = \frac{1}{2} \log \frac{q}{\pi} + \frac{1}{2} \frac{\Gamma'}{\Gamma}((s+\varepsilon)/2) - \sum_{\rho} \operatorname{Re} \frac{1}{s-\rho}.$$

Here  $\rho$  runs over the nontrivial zeros with multiplicity. The sum of real parts converges, and each term is nonnegative. This is the standard logarithmic-derivative identity; see [5, Sections 12 and 14]. The functional equation cancels the real constant from the Hadamard product. For  $1 < s \leq 2$ , the gamma term is bounded for both parities. Keeping only the term for  $\beta$  and using  $-\zeta'/\zeta(s) = 1/(s-1) + O(1)$  gives

$$(2.4) \quad -\frac{\zeta'}{\zeta}(s) - \frac{L'}{L}(s, \chi) \leq C\ell + \frac{1}{s-1} - \frac{1}{s-\beta}.$$

Take  $s = 1 + 1/\log X$ , which lies in  $(1, 2)$  for  $X \geq 3$ . Writing  $\Lambda(n)$  for the von Mangoldt function, the Euler products give the nonnegative series

$$-\frac{\zeta'}{\zeta}(s) - \frac{L'}{L}(s, \chi) = \sum_{n \geq 1} \frac{\Lambda(n)(1 + \chi(n))}{n^s} \geq \frac{2}{e} \sum_{\substack{p \leq X \\ \chi(p)=1}} \frac{\log p}{p}.$$

Also,

$$\frac{1}{s-1} - \frac{1}{s-\beta} = \frac{1-\beta}{(s-1)(s-\beta)} \leq (1-\beta)(\log X)^2.$$

This proves (2.1). To obtain (2.2), use Mertens' estimate and the elementary bound for primes dividing  $q$ :

$$\sum_{p \leq X} \frac{\log p}{p} = \log X + O(1), \quad \sum_{p|q} \frac{\log p}{p} \leq \log q.$$

Removing primes at most  $H$  costs a constant depending only on  $H$ . □

## 3. INTERPOLATION ON A PROJECTED INTEGER BOX

The purpose of this section is to obtain a nonzero evaluation determinant using monomials whose separate degrees may be very different. No arithmetic estimates enter the interpolation lemma.

**Lemma 3.** *Let  $A : \mathbb{C}^4 \rightarrow \mathbb{C}^3$  be a surjective linear map, with  $\ker A = \mathbb{C}c$ , where the four coordinates of  $c$  are linearly independent over  $\mathbb{Q}$ . Let  $N \geq 1$  and  $t_1, t_2, t_3$  be integers satisfying*

$$(3.1) \quad t_j \geq 3(N-1) \quad (1 \leq j \leq 3), \quad \prod_{j=1}^3 (t_j - 3(N-1) + 1) > (4N-3)^4.$$

Set  $E_N = \{0, \dots, N-1\}^4$  and

$$\mathcal{P}_t = \{B \in \mathbb{C}[z_1, z_2, z_3] : \deg_{z_j} B \leq t_j \text{ for } 1 \leq j \leq 3\}.$$

Then the evaluation map  $B \mapsto (B(An))_{n \in E_N}$  from  $\mathcal{P}_t$  to  $\mathbb{C}^{E_N}$  is surjective.

*Proof.* Suppose the evaluation map is not surjective. There are complex numbers  $v_n$ , not all zero, such that

$$(3.2) \quad \sum_{n \in E_N} v_n B(An) = 0 \quad (B \in \mathcal{P}_t).$$

Let  $S = \{n : v_n \neq 0\}$  and  $P = \text{conv}(S) \subset \mathbb{R}^4$ . We construct the test polynomial as a product. A translated polynomial will vanish at  $An$  for every  $n \in S$  outside one supporting face of  $P$ . A second polynomial will select a single vertex on that face, with total degree at most  $3(N-1)$ .

Put  $\mathcal{K} = 4P$ . For  $b_j = t_j - 3(N-1)$ , the polynomials of separate degrees at most  $b_j$  form a vector space of dimension  $\prod_j (b_j + 1)$ . Since

$$\mathcal{K} \subset [0, 4(N-1)]^4, \quad \#(\mathcal{K} \cap \mathbb{Z}^4) \leq (4N-3)^4 < \prod_j (b_j + 1),$$

there is a nonzero polynomial  $R$  with  $\deg_{z_j} R \leq b_j$  and

$$(3.3) \quad R(Am) = 0 \quad (m \in \mathcal{K} \cap \mathbb{Z}^4).$$

*A supporting face.* The polynomial  $R \circ A$  is nonzero because  $A$  is surjective. A nonzero polynomial on  $\mathbb{C}^4$  cannot vanish on all of  $\mathbb{Z}^4$ : this follows by applying the one-variable root bound successively to its four coordinates. Choose  $m \in \mathbb{Z}^4$  with  $R(Am) \neq 0$  whose distance to  $\mathcal{K}$  is least. Such a minimum exists because bounded neighborhoods of the compact set  $\mathcal{K}$  contain finitely many lattice points. Let  $y \in \mathcal{K}$  be nearest to  $m$  and set  $h = m - y$ . By (3.3),  $h \neq 0$ .

The nearest-point inequality  $h \cdot (x - y) \leq 0$  for  $x \in \mathcal{K}$  shows that  $y/4$  belongs to the supporting face

$$G = \{v \in P : h \cdot v = \max_{w \in P} h \cdot w\}.$$

This face has dimension at most three: if  $P$  is four-dimensional then  $G$  is proper, and otherwise  $\dim G \leq \dim P \leq 3$ . Removing affine dependencies from a convex combination expresses  $y/4$  as a combination of at most four vertices of  $G$ . One coefficient is at least  $1/4$ . Choosing its vertex  $u$  gives

$$(3.4) \quad u \in S \cap G, \quad y - u \in 3P.$$

Indeed, subtracting  $u$  from four times that convex combination leaves nonnegative coefficients with sum three.

For  $n \in S \setminus G$ , define

$$y' = y + n - u \in 4P, \quad m' = m + n - u = y' + h.$$

The inclusion follows from (3.4). Since  $h \cdot (u - n) > 0$ , for sufficiently small  $\eta > 0$  the point  $y' + \eta(y - y') \in \mathcal{K}$  satisfies

$$(3.5) \quad \|m' - [y' + \eta(y - y')]\|^2 = \|h - \eta(u - n)\|^2 < \|h\|^2.$$

Thus  $m'$  is closer to  $\mathcal{K}$  than  $m$  is. By the choice of  $m$ ,

$$(3.6) \quad R(A(m+n-u)) = 0 \quad (n \in S \setminus G).$$

Figure 1 illustrates the strict decrease in distance.

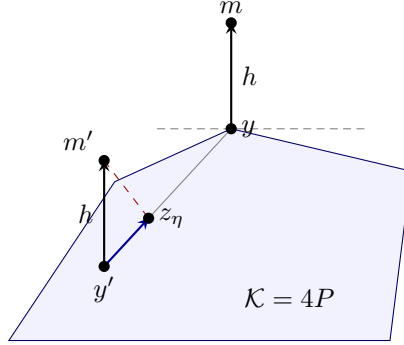


FIGURE 1. Schematic of (3.5). Both  $y$  and  $y'$  lie in  $\mathcal{K}$ , and  $m - y = m' - y' = h$ . Moving to  $z_\eta = y' + \eta(y - y')$  decreases the distance to  $m'$  because  $h \cdot (y - y') > 0$ . The dashed red segment is shorter than  $\|h\|$ ; the argument takes place in  $\mathbb{R}^4$ .

*Interpolation on the face.* The integer vertices of  $G$  span an affine space of dimension at most three. Solving their affine linear equations over  $\mathbb{Q}$  and clearing denominators gives a hyperplane

$$\mathcal{H} = \{v \in \mathbb{C}^4 : r \cdot v = k\}, \quad 0 \neq r \in \mathbb{Z}^4, \quad k \in \mathbb{Z},$$

containing  $G$ . Rational independence gives  $r \cdot c \neq 0$ . The restriction  $A|_{\mathcal{H}}$  is therefore an affine bijection onto  $\mathbb{C}^3$ : each affine line parallel to  $\ker A$  meets  $\mathcal{H}$  exactly once.

Choose  $i_0$  with  $r_{i_0} \neq 0$ . The other three coordinates determine a point of  $\mathcal{H}$ . Let  $\lambda_i(z)$ , for  $i \neq i_0$ , be those affine coordinate functions of  $(A|_{\mathcal{H}})^{-1}(z)$ . Thus  $\lambda_i(An) = n_i$  for  $n \in \mathcal{H}$ . The Lagrange polynomial

$$Q(z) = \prod_{i \neq i_0} \prod_{\substack{0 \leq a < N \\ a \neq u_i}} \frac{\lambda_i(z) - a}{u_i - a}$$

has total degree at most  $3(N-1)$  and satisfies

$$Q(An) = \begin{cases} 1, & n = u, \\ 0, & n \in (S \cap G) \setminus \{u\}. \end{cases}$$

The denominators are nonzero integers; empty products are 1.

Now set  $B(z) = Q(z)R(z + A(m-u))$ . Translation preserves each separate degree of  $R$ , and the total degree bound on  $Q$  bounds each of its separate degrees. Hence  $B \in \mathcal{P}_t$ . Equation (3.6) eliminates the terms off  $G$ , and  $Q$  eliminates all remaining terms except the one at  $u$ . Substitution in (3.2) gives the contradiction

$$0 = \sum_{n \in S} v_n Q(An) R(A(m+n-u)) = v_u R(Am) \neq 0. \quad \square$$

**Corollary 4.** *Let  $A$  satisfy the hypotheses of Lemma 3. For integers  $1 \leq H \leq N$ , set*

$$U = N^{4/3}, \quad T_1 = 32H^{2/3}U, \quad T_2 = T_3 = 32H^{-1/3}U.$$

*The rows  $((An)_1^{\alpha_1} (An)_2^{\alpha_2} (An)_3^{\alpha_3})_{n \in E_N}$ , with  $\alpha \in \mathbb{Z}_{\geq 0}^3$  and  $\alpha_j \leq T_j$ , span  $\mathbb{C}^{E_N}$ . The constant 32 is independent of  $A$  and  $H$ .*

*Proof.* Put  $t_j = \lfloor T_j \rfloor$ . Since  $H \leq N$ , every  $T_j \geq 32N$ , and  $T_1 T_2 T_3 = 32^3 N^4$ . In particular  $t_j \geq 3(N-1)$ , and

$$\prod_{j=1}^3 (t_j - 3(N-1) + 1) > \prod_{j=1}^3 (T_j - 3(N-1)) \geq \frac{T_1 T_2 T_3}{8} = 4096 N^4 > (4N-3)^4.$$

Apply Lemma 3 to the monomial basis of  $\mathcal{P}_t$ .  $\square$

*Remark 5.* The connection with multiplicity estimates is immediate. On  $(\mathbb{C}^\times)^4$ , write  $D_j = \sum_i A_{ji} x_i \partial / \partial x_i$ . For  $x^n = x_1^{n_1} \cdots x_4^{n_4}$ ,  $(D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3} x^n)(1, 1, 1, 1) = (An)^\alpha$ . Thus Corollary 4 is a rectangular multiplicity estimate. This interpolation viewpoint is standard in the subject; compare [6, Section 1]. We use the evaluation rows directly in what follows.

#### 4. A WEIGHTED DETERMINANT OF CONJUGATES

We now construct a nonzero determinant for which most of the total exponent lies in the first coordinate. This is the coordinate to which the prime congruence will apply.

A primitive nonprincipal real character  $\chi$  corresponds to a fundamental discriminant  $D$  with  $|D| = q$ . Write  $\mathbb{Q}(\sqrt{D}) = \mathbb{Q}(\sqrt{d})$  with  $d$  squarefree. Then  $|d| \leq q$  and, for odd  $p \nmid q$ ,  $\chi(p) = (d/p)$ , the Legendre symbol; see [5]. Temporarily exclude  $d = 2$  and put

$$a = \sqrt{d}, \quad b = \sqrt{2}, \quad K = \mathbb{Q}(a, b), \quad \mathcal{R} = \mathbb{Z}[a, b].$$

Since  $\chi$  is nonprincipal,  $d \neq 1$ , so  $[K : \mathbb{Q}] = 4$ . Every element of  $\mathcal{R}$  has a unique expression  $z_1 + z_2 a + z_3 b + z_4 ab$  with  $z_i \in \mathbb{Z}$ . The automorphisms  $\sigma(a) = -a$ ,  $\sigma(b) = b$  and  $\tau(a) = a$ ,  $\tau(b) = -b$  generate  $\text{Gal}(K/\mathbb{Q})$ .

For integers  $1 \leq H \leq N$ , retain the notation  $U = N^{4/3}$  and set  $M = N^4 = U^3$ . For  $n \in E_N$  define  $\theta_n = n_1 + n_2 a + n_3 b + n_4 ab$ . The map

$$(4.1) \quad A = \begin{pmatrix} 1 & a & b & ab \\ 1 & -a & b & -ab \\ 1 & -a & -b & ab \end{pmatrix} \quad \text{satisfies} \quad An = (\theta_n, \sigma(\theta_n), \sigma\tau(\theta_n)).$$

It has rank three and

$$\ker A = \mathbb{C}(ab, b, -a, -1).$$

For example, the minor in the first three columns is  $4ab \neq 0$ , and multiplication verifies the displayed kernel. Its four coordinates are linearly independent over  $\mathbb{Q}$  because  $1, a, b, ab$  are a  $\mathbb{Q}$ -basis of  $K$ . Thus Corollary 4 applies.

For  $\alpha \in \mathbb{Z}_{\geq 0}^3$ , write

$$(4.2) \quad R_\alpha = (\theta_n^{\alpha_1} \sigma(\theta_n)^{\alpha_2} \sigma\tau(\theta_n)^{\alpha_3})_{n \in E_N}.$$

Fix an order of the columns. Order all indices in  $\mathbb{Z}_{\geq 0}^3$  by increasing  $w(\alpha) = \alpha_1 + H\alpha_2 + H\alpha_3$ , with any fixed rule to break ties. Retain a row precisely when it increases the span of the earlier rows over  $K$ . Rank is unchanged on extending scalars from  $K$  to  $\mathbb{C}$ , so Corollary 4 guarantees  $M$  retained rows. Every row in that corollary has weight at most  $T_1 + HT_2 + HT_3 = 96H^{2/3}U$ . Therefore all retained rows have weight at most this bound.

Let  $\mathcal{P}$  be the set of retained indices, in their retained order, and put

$$(4.3) \quad \Delta = \det(R_\alpha)_{\alpha \in \mathcal{P}} \in \mathcal{R} \setminus \{0\}, \quad S_1 = \sum_{\alpha \in \mathcal{P}} \alpha_1, \quad S_2 = \sum_{\alpha \in \mathcal{P}} (\alpha_2 + \alpha_3).$$

Only the surjectivity conclusion of Corollary 4 enters the selection of these rows. The coefficients of the auxiliary polynomials used to prove it may depend on  $A$ , but they do not enter  $\Delta$ , which is formed from the original monomial rows. Thus no bound on those coefficients is needed for the size estimate below. This is the interpolation-determinant construction, with a weighted row order; compare [11, Section 6].

**Lemma 6.** *There are absolute constants  $c_0, C_0 > 0$  such that for every fixed integer  $H \geq 1$  and all sufficiently large  $N$  in terms of  $H$  alone,*

$$(4.4) \quad S_1 \geq c_0 M H^{2/3} U, \quad S_2 \leq C_0 M H^{-1/3} U.$$

*In particular,  $S_2/S_1 \leq (C_0/c_0)/H$ .*

*Proof.* The weight bound gives  $\alpha_2, \alpha_3 \leq B := 96H^{-1/3}U$  for every retained index. Thus  $S_2 \leq 192M H^{-1/3}U$ . For each fixed value of  $\alpha_1$ , there are at most

$$Q_0 = (\lfloor B \rfloor + 1)^2 \leq 97^2 H^{-2/3} U^2$$

retained indices; here  $H^{-1/3}U \geq 1$  follows from  $N \geq H$ . List the first coordinates in increasing order as  $b_1, \dots, b_M$ . Then  $b_i \geq \lfloor (i-1)/Q_0 \rfloor$ . Writing  $M = kQ_0 + r$ ,  $0 \leq r < Q_0$ , gives

$$S_1 \geq Q_0 \frac{k(k-1)}{2} + kr \geq \frac{M^2}{2Q_0} - \frac{M}{2}.$$

The last inequality has difference  $r(Q_0 - r)/(2Q_0) \geq 0$ . For fixed  $H$ , the condition  $M \geq 2Q_0$  holds whenever  $H^{2/3}N^{4/3} \geq 2 \cdot 97^2$ . Thus, uniformly in the field,

$$S_1 \geq \frac{M^2}{4Q_0} \geq \frac{MH^{2/3}U}{4 \cdot 97^2}.$$

We may take  $c_0 = (4 \cdot 97^2)^{-1}$  and  $C_0 = 192$ . □

## 5. TWO BOUNDS FOR THE DETERMINANT

For  $z \in K$ , define  $N(z) = z\sigma(z)\tau(z)\sigma\tau(z)$ . If  $z \in \mathcal{R}$ , then  $N(z) \in \mathbb{Z}$ : its product belongs to  $\mathcal{R}$  and is fixed by both  $\sigma$  and  $\tau$ , so only its integer constant coefficient remains. In particular  $N(\Delta)$  is a nonzero integer. We bound its absolute value from above at the complex embeddings and from below by divisibility.

**The Archimedean bound.** For every embedding  $\nu : K \hookrightarrow \mathbb{C}$  and every  $n \in E_N$ ,

$$|\nu(\theta_n)| \leq (N-1)(1 + \sqrt{|d|})(1 + \sqrt{2}) \leq 8N\sqrt{q}.$$

This estimate is valid for either sign of  $d$ . Each row indexed by  $\alpha$  has Euclidean norm at most  $\sqrt{M}(8N\sqrt{q})^{\alpha_1+\alpha_2+\alpha_3}$  at every embedding. Hadamard's inequality and the four embeddings give

$$(5.1) \quad \frac{1}{4} \log |N(\Delta)| \leq \frac{M}{2} \log M + (S_1 + S_2)(\log N + \frac{1}{2}\ell + \log 8).$$

**The prime divisibility.** Call a prime  $p$  admissible if  $p > H$ ,  $p \nmid 2q$ , and  $\chi(p) = -1$ . Euler's criterion gives in  $\mathcal{R}/p\mathcal{R}$

$$a^p = a d^{(p-1)/2} \equiv -a, \quad b^p = b 2^{(p-1)/2} \equiv (2/p)b.$$

The  $p$ -th-power map is a ring homomorphism in characteristic  $p$ , and integer coefficients are fixed by it. Consequently

$$(5.2) \quad \theta^p \equiv g_p(\theta) \pmod{p\mathcal{R}} \quad (\theta \in \mathcal{R}), \quad g_p = \begin{cases} \sigma, & (2/p) = 1, \\ \sigma\tau, & (2/p) = -1. \end{cases}$$

This is the Frobenius relation in the ring  $\mathcal{R}$ .

**Lemma 7.** *For every admissible prime  $p$ ,*

$$(5.3) \quad \Delta \in p^{E_p}\mathcal{R}, \quad E_p = \sum_{\alpha \in \mathcal{P}} \left\lfloor \frac{\alpha_1}{p} \right\rfloor.$$

*Proof.* Let  $j = 2$  or  $3$  according as  $g_p = \sigma$  or  $\sigma\tau$ . For a retained index  $\alpha$ , write  $\alpha_1 = pk + r$  with  $0 \leq r < p$ . In (4.2), replace the first factor by

$$\theta_n^r (\theta_n^p - g_p(\theta_n))^k$$

and leave the other two factors unchanged. Denote the resulting row by  $\tilde{R}_\alpha$ . Expansion yields the exact identity

$$(5.4) \quad \tilde{R}_\alpha = R_\alpha + \sum_{m=1}^k (-1)^m \binom{k}{m} R_{\alpha - pme_1 + me_j},$$

where  $e_1, e_2, e_3$  are the standard coordinate vectors. Every row on the right after  $R_\alpha$  has smaller weight, since its weight is  $w(\alpha) - m(p - H)$ .

By the greedy selection rule, every earlier row lies in the  $K$ -span of the earlier retained rows. Thus the matrix of replacement rows equals  $L$  times the matrix of retained rows, where  $L$  is lower triangular over  $K$  with diagonal entries 1. In particular,

$$(5.5) \quad \det(\tilde{R}_\alpha)_{\alpha \in \mathcal{P}} = \Delta.$$

On the other hand, (5.2) shows that every entry of  $\tilde{R}_\alpha$  belongs to  $p^k \mathcal{R}$ . Dividing its entries by  $p^k$  leaves a matrix over  $\mathcal{R}$ . Factoring these powers from (5.5) proves (5.3). The equality of determinants is over  $K$ ; the divisibility follows from the replacement entries themselves, so no integrality of  $L$  is required.  $\square$

Taking norms in (5.3) gives  $p^{4E_p} \mid N(\Delta)$ . Since the norm is a nonzero integer, distinct admissible primes may be combined. Therefore

$$(5.6) \quad \begin{aligned} \frac{1}{4} \log |N(\Delta)| &\geq \sum_{\substack{p \leq U \\ p \text{ admissible}}} E_p \log p \\ &\geq S_1 \sum_{\substack{p \leq U \\ p \text{ admissible}}} \frac{\log p}{p} - M \sum_{p \leq U} \log p. \end{aligned}$$

The second inequality uses  $\lfloor x \rfloor \geq x - 1$  in each row. Chebyshev's bound  $\sum_{p \leq U} \log p \ll U$  and Lemma 2 now give, when  $U \geq 3$  and  $H \geq 2$ ,

$$(5.7) \quad \frac{1}{4} \log |N(\Delta)| \geq S_1 \left( \log U - C\ell - C \frac{\delta(\log U)^2}{\ell} - C_H \right) - CMU.$$

The leading term here is  $S_1 \log U$ . In (5.1), the corresponding term is  $(S_1 + S_2) \log N$ , and  $\log N = \frac{3}{4} \log U$ . The weight  $H$  makes  $S_2/S_1$  small enough to preserve this difference.

## 6. COMPLETION OF THE PROOF

Suppose Theorem 1 is false. Then there is a sequence of primitive nonprincipal real characters and real zeros with

$$(6.1) \quad q \longrightarrow \infty, \quad \delta = (1 - \beta) \log q \longrightarrow 0.$$

To justify  $q \rightarrow \infty$ , there are only finitely many characters of bounded conductor, and for each of them  $L(1, \chi) \neq 0$ . Analyticity therefore prevents zeros from accumulating at 1. Passing to a subsequence gives (6.1). In particular  $\mathbb{Q}(\sqrt{d}) \neq \mathbb{Q}(\sqrt{2})$  for all sufficiently large  $q$ , so the field construction applies.

Fix  $H \geq 2$  for the moment. Combining (5.1) and (5.7), then dividing by  $S_1 \log U$ , yields

$$(6.2) \quad \begin{aligned} 1 &\leq \frac{3}{4} \left( 1 + \frac{S_2}{S_1} \right) + \left( C + \frac{1}{2} \left( 1 + \frac{S_2}{S_1} \right) \right) \frac{\ell}{\log U} + C\delta \frac{\log U}{\ell} \\ &\quad + \frac{C_H + (1 + S_2/S_1) \log 8}{\log U} + \frac{CMU + \frac{1}{2}M \log M}{S_1 \log U}. \end{aligned}$$

The constants denoted by  $C$  are absolute. The division is legitimate for all sufficiently large  $N$  by Lemma 6.

First choose the integer  $H$  so large that  $(C_0/c_0)/H \leq 1/12$ . Lemma 6 then makes the first term of (6.2) at most  $13/16$ .

Next choose a fixed real  $\gamma > 0$  and set  $N = \lceil q^\gamma \rceil$ . Along (6.1),

$$\frac{\log U}{\ell} \longrightarrow \frac{4\gamma}{3}.$$

Take  $\gamma$  sufficiently large that the limiting upper bound for the second term of (6.2) is less than  $1/16$ . This choice depends only on the absolute constants and the already fixed bound  $S_2/S_1 \leq 1/12$ .

Finally let  $q \rightarrow \infty$  along (6.1). The third term tends to zero because  $\gamma$  is fixed and  $\delta \rightarrow 0$ . The fourth tends to zero because  $H$  is fixed. For the last term, Lemma 6 and  $\log M = 3 \log U$  give

$$\frac{MU}{S_1 \log U} \ll \frac{H^{-2/3}}{\log U}, \quad \frac{M \log M}{S_1 \log U} \ll \frac{H^{-2/3}}{U},$$

both tending to zero. The conditions  $N \geq H$ ,  $U \geq 3$ , and the lower bound on  $S_1$  all hold eventually. Taking an upper limit in (6.2) gives  $1 \leq 13/16 + 1/16 = 7/8$ , a contradiction. This proves Theorem 1.

## REFERENCES

- [1] K. Barreto, *An elementary proof of the non-existence of Landau–Siegel zeros*, manuscript (2026), posted at <https://x.com/acerfur/status/2112239827077026128>.
- [2] W. D. Banks, *On the OpenAI proof that Landau–Siegel zeros do not exist*, preprint (2026). [arXiv:2610.15483](https://arxiv.org/abs/2610.15483).
- [3] K. Benli, S. Goel, H. Twiss, and A. Zaman, *Explicit Deuring–Heilbronn phenomenon for Dirichlet  $L$ -functions*, Proc. Amer. Math. Soc. **154** (2026), no. 2, 509–525. doi:10.1090/proc/17450.
- [4] J.-B. Bost, *Algebraic leaves of algebraic foliations over number fields*, Publ. Math. Inst. Hautes Études Sci. **93** (2001), 161–221. doi:10.1007/s10240-001-8191-3.
- [5] H. Davenport, *Multiplicative number theory*, 2nd ed., revised by H. L. Montgomery, Graduate Texts in Mathematics, vol. 74, Springer-Verlag, New York, 1980.
- [6] S. Fischler, *Interpolation on algebraic groups*, Compos. Math. **141** (2005), no. 4, 907–925. doi:10.1112/S0010437X05001351.
- [7] J. B. Friedlander and H. Iwaniec, *A note on Dirichlet  $L$ -functions*, Expo. Math. **36** (2018), nos. 3–4, 343–350. doi:10.1016/j.exmath.2018.06.003.
- [8] B. Green, *On Landau–Siegel zeros*, preprint (2026). [hexagon:2610.00311](https://arxiv.org/abs/2610.00311).
- [9] D. R. Heath-Brown, *Zero-free regions for Dirichlet  $L$ -functions, and the least prime in an arithmetic progression*, Proc. London Math. Soc. (3) **64** (1992), no. 2, 265–338. doi:10.1112/plms/s3-64.2.265.
- [10] B. Kerr, *A simpler proof that there are no Landau–Siegel zeros*, preprint (2026). [arXiv:2610.21099](https://arxiv.org/abs/2610.21099).
- [11] M. Laurent, *Sur quelques résultats récents de transcendance*, Astérisque **198–199–200** (1991), 209–230. [https://www.numdam.org/item/AST\\_1991\\_\\_198-199-200\\_209\\_0/](https://www.numdam.org/item/AST_1991__198-199-200_209_0/).
- [12] U. V. Linnik, *On the least prime in an arithmetic progression. II. The Deuring–Heilbronn phenomenon*, Rec. Math. [Mat. Sbornik] N.S. **15(57)** (1944), no. 3, 347–368. <https://www.mathnet.ru/eng/sm6202>.
- [13] R. F. Lu, A. Zaman, and H. Zhao, *Numerical computations concerning Landau–Siegel zeros*, preprint (2026). [arXiv:2602.03626](https://arxiv.org/abs/2602.03626).
- [14] D. W. Masser, *On polynomials and exponential polynomials in several complex variables*, Invent. Math. **63** (1981), no. 1, 81–95. doi:10.1007/BF01389194.
- [15] A. Page, *On the number of primes in an arithmetic progression*, Proc. London Math. Soc. (2) **39** (1935), 116–141. doi:10.1112/plms/s2-39.1.116.
- [16] J. Pescador, private communication (2026).
- [17] P. Philippon, *Lemmes de zéros dans les groupes algébriques commutatifs*, Bull. Soc. Math. France **114** (1986), 355–383; errata et addenda, **115** (1987), 397–398. doi:10.24033/bsmf.2060; doi:10.24033/bsmf.2084.
- [18] C. L. Siegel, *Über die Classenzahl quadratischer Zahlkörper*, Acta Arith. **1** (1935), 83–86. doi:10.4064/aa-1-1-83-86.
- [19] T. Tao, *A sketch of the proof of non-existence of Siegel zeroes*, blog post, *What’s new*, October 19, 2026. <https://terrytao.wordpress.com/2026/10/19/a-sketch-of-the-proof-of-non-existence-of-siegel-zeroes/>.